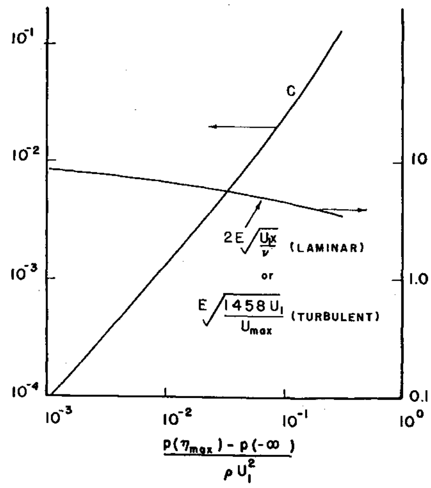

Fig. 3 Effect of lateral pressure difference on curvature and on entrainment.

boundary the velocity profile, and therefore f'^2 varies very little with curvature the integral on the right-hand side of Eq. (18) is only slightly affected by the curvature. The curvature parameter C, and therefore the curvature, varies then almost directly with the pressure difference across the freejet boundary. This is shown in Fig. 3 where the curvature parameter is shown as a function of the normalized pressure difference. As expected increasing the pressure difference increases the curvature of the flow.

Sawyer[4] in a study of reattaching jet flows, defined an entrainment parameter, E, which is a measure of the rate at which fluid is entrained by the flow. The fluid entrainment is important physically in determining the extent of the reattachment region in the Coanda effect. The entrainment parameter is defined by

$$E = \frac{1}{U_1}\frac{d}{dx}\int_{-\infty}^{y_{\max}} u dy$$

where $y_{\max}$ is the value of y corresponding to the joining of the freejet boundary and the irrotational flow. In the present analysis, one obtains for a normalized entrainment parameter in the case of laminar flow

$$2E(U_1x/\nu)^{1/2} = f(\eta_{\max}) - f(-\infty)$$

and in the case of turbulent flow

$$E(1458U_1/U_{\max})^{1/2} = f(\eta_{\max}) - f(-\infty)$$

These parameters are also presented in Fig. 3 as a function of the normalized pressure difference. It is particularly interesting to note that as the pressure difference across the jet boundary is increased the curvature of the jet boundary increases substantially while the entrainment changes only slightly and in fact decreases. The decrease is due to the alteration of the velocity profile near $\eta_{\max}$.

In the case of the curved freejet boundary then the curvature of jet is strongly influenced by the pressure difference across the jet while the entrainment is relatively insensitive to this pressure difference.

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A Matched Asymptotic Solution for Skipping Entry into Planetary Atmosphere

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I. Introduction

IN a previous paper (Ref. 1), an asymptotic solution was obtained for lifting trajectories entering a planetary atmosphere. The solution was derived by use of the method of matched asymptotic expansions, and in contrast to Refs. 2 and 3, employed only two regions: an "inner region" inside the atmosphere with dominating aerodynamic forces, and an "outer region" where gravitational forces dominate. Solutions for the "inner" and "outer" regions were matched resulting in a "composite solution" valid in both regions.

In Ref. 4, Loh classified entry trajectories as a) ballistic, b) gliding, c) skipping, and d) oscillatory paths and gave a detailed analysis of oscillatory paths. This paper is concerned with "skip paths" which are oscillatory paths of large oscillation amplitudes characterized by the fact that the vehicle leaves and re-enters the atmosphere several times before final entry. Skipping trajectories cannot be treated by the usual techniques such as those of Refs. 4 and 5, but require application of solutions which are valid both inside and outside the atmosphere. This requirement is fulfilled by the composite solution presented in Ref. 1.

The goals of this paper are 1) to show that the asymptotic approach developed by Shi and Pottsepp[1] can be extended to skipping entry trajectories, 2) to demonstrate the accuracy of results by comparing the matched asymptotic solution with the exact numerical integrations, and 3) to show that a simple closed form analytic composite solution which is uniformly valid to order unity everywhere can be used to calculate the skipping entry trajectories up to the fourth or fifth extremal points with surprisingly good accuracy. Only terms of order unity are considered in the present study.

II. Skipping Entry

Results of Ref. 1 can be extended to skipping trajectories by observing that the composite solution remains valid over

Table 1 Comparison of asymptotic expansion and numerical integrations—skipping entry

Pt #	Numerical integration I (exponential atmosphere)		Numerical integration II (isothermal atmosphere)		Asymptotic expansion (with improved constants)	
	$V'(gR)^{-1/2}$	h', ft	$V'(gR)^{-1/2}$	h', ft	$V'(gR)^{-1/2}$	h', ft
0	1.3987	360000	1.3987	360000	1.3987	360000
1	1.0378	207810	1.0374	210500	1.0325	207913
2	0.6890	1466161	0.6885	1468770	0.6839	1386558
3	0.5585	207070	0.5581	209611	0.5506	207913
4	0.3799	405842	0.3791	408254	0.3744	382755
5	0.2778	203757	0.2757	206111	0.2764	207913

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each ascending or descending arc of a skipping trajectory, with the exception that integration constants associated with each arc need to be determined in a slightly different manner.

Using the notation of Ref. 1, the composite solution may be written in the form

$$ {}_nq_c = {}_nV_{00}^2 \exp\{-(2C_D/C_L)({}_n\gamma_0^* - {}_n\gamma_{00}^*)\} - 2h/r \quad (1)$$

$$\cos{}_n\gamma_c = {}_nV_{00}^2 \cos{}_n\gamma_{00}/(r)^{1/2}({}_nV_{00}^2r - 2h)^{1/2} + \tfrac{1}{2}C_L\rho(0)e^{-h^*} \quad (2)$$

$$\cos{}_n\gamma_0^* = \cos{}_n\gamma_{00}^* + \tfrac{1}{2}C_L\rho(0)e^{-h^*} \quad (3)$$

where $r = 1 + h$ and $n = 1,2,3, \ldots N$ identifies the arc under consideration. On the first arc ($n = 1$), constants ${}_1\gamma_{00}$ and ${}_1V_{00}^2$ can be determined from Eqs. (19) and (20) of Ref. 1, provided that the trajectory starts in the outer region. In general, ${}_1\gamma_{00}$ and ${}_1\gamma_{00}^*$ are related by matching. However, for a skipping trajectory, determination of ${}_1\gamma_{00}^*$ is more involved. Considering a skip trajectory shown in Figs. 1 and 2, we notice that the flight path angle γ vanishes at the extremal points 1, 2, 3, 4, . . . i.e., γ_c changes quadrants whenever $\cos\gamma_c$ reaches unity. In general, γ_0^* will not vanish at the same point as γ_c because they differ in terms of order ϵ. In fact, γ_0^* may not always be defined at all at $\gamma_c = 0$ because s_0^* may turn out to be larger than unity at that point. The discontinuity in the expression of γ_0^* causes errors in determining V_c. Furthermore, the errors may grow after each cycle because of the phase difference in γ_0^* and γ_c^* at the point $\gamma_c^* = 0$.

However, it should be remembered that Eqs. (2) and (3) are correct only to order unity and at the first minimum point, s_0^* and $\cos\gamma_c$ differ only by terms of order ϵ. Also, the constants $\cos\gamma_{00}$ and V_{00} are determined only to order unity for a solution uniformly valid to order unity everywhere. Thus, these constants may be changed by quantities to order ϵ without increasing the error of Eqs. (2) and (3). In this way, it is possible to avoid the aforementioned difficulties by adjusting s_{00}^* in the inner solution such that γ_0^* and γ_c vanish simultaneously. Thus, the constant ${}_1\gamma_{00}^*$ is determined by letting

$$ {}_1\gamma_c = {}_1\gamma_0^* = 0 \quad (4)$$

at the first minimum point "1."

In Fig. 3, which shows the first descending arc of Fig. 1, the dashed curve corresponds to the trajectory calculated by using the matching conditions of Ref. 1, and the solid curve corresponds to the case obtained from Eqs. (1–3) by adjusting ${}_1\gamma_{00}$ so that γ_0^* and γ_c vanish simultaneously at the minimum point. It is clear from Fig. 1 that the error between them is indeed of high order.

Since the second arc starts in the inner region (the minimum point "1"), Eqs. 19 and 20 of Ref. 1 cannot be used for determining ${}_2V_{00}$ and ${}_2\gamma_{00}$, and the latter are determined by requiring continuity of the composite solution and $\cos\gamma_0^*$ at $h = h_1$.

Since ${}_2\gamma_0^*$ changes sign at "1," Eqs. (1–3) give

$$ {}_2\gamma_{00}^* = -{}_1\gamma_{00}^* \quad (5)$$

$$ {}_2V_{00}^2 = {}_1V_{00}^2 \exp[(4C_D/C_L){}_1\gamma_{00}^*] \quad (6)$$

Table 2 Comparison of asymptotic expansion and numerical integration—oscillatory entry

	Numerical integration			Asymptotic expansion		
Pt #	$V'(gR)^{-1/2}$	h', ft	$\beta h'$	$V'(gR)^{-1/2}$	h', ft	$\beta h'$
0	1.00	360000	15.32	1.00	360000	15.32
1	0.8139	225001	9.57	0.8121	225412	9.59
2	0.6298	564119	24.01	0.6290	563558	23.98
3	0.5184	223342	9.50	0.5231	225410	9.59
4	0.3903	324668	13.82	0.4059	319653	13.60

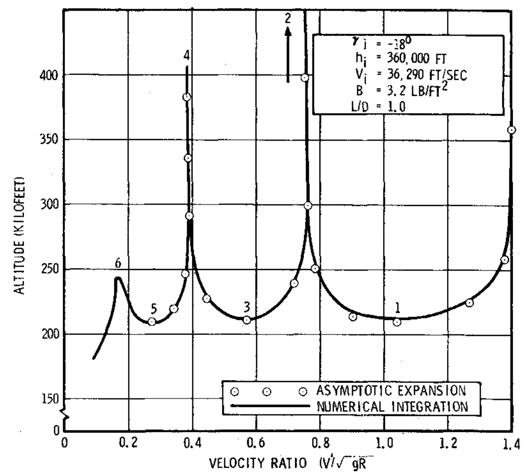

Fig. 1 Altitude vs velocity ratio—skipping entry.

$$\cos{}_2\gamma_{00} = [1 - (C_L/2)\rho(0)\exp(-h_1^*)]r_1^{1/2} \times \{r_1 - 2h_1/{}_2V_{00}^2\}^{1/2} \quad (7)$$

For the numerical example considered in Figs. 1 and 2, the first peak point or point "2" is about 1,400,000 ft or 250 miles above the surface of the Earth (too high to be shown in Fig. 1) and is clearly in the outer region. It may be pointed out that no previous solution was capable of adequately describing the portion of trajectory between the minimum point "1" and the maximum point "2."

Constants associated with the remainder of the arcs are evaluated in an analogous fashion. It is not expected that the present composite solution will be applicable after the minimum point "5" because the last peak is entirely inside the atmosphere, and the trajectory no longer skips to the outer region. It is believed that the last portion of the trajectory may be treated by inner solutions of higher order discussed in Ref. 1.

III. Comparison with Numerical Results

The aforementioned procedure was first applied to the case, $\gamma_i = -18°$, $h_i' = 360{,}000$ ft, $V_i' = 36{,}290$ fps, $B = 3.2$ lb/ft^2 and $L/D = 1$. The results are plotted in Figs. 1, 2, and 3 and tabulated in Table 1. The quality of agreement with numerical results demonstrates the validity of application of the matched asymptotic methods to the present problem.

We also studied the case corresponding to an "oscillatory path" considered by Loh[4,5] with $\gamma_i = -12°$, $V_i' = 25{,}945$ ft, $h_i' = 360{,}000$ ft and $L/D = 1$ to obtain an idea of how accurate our asymptotic solution is when applied to oscillatory motion. Comparison of matched asymptotic solution

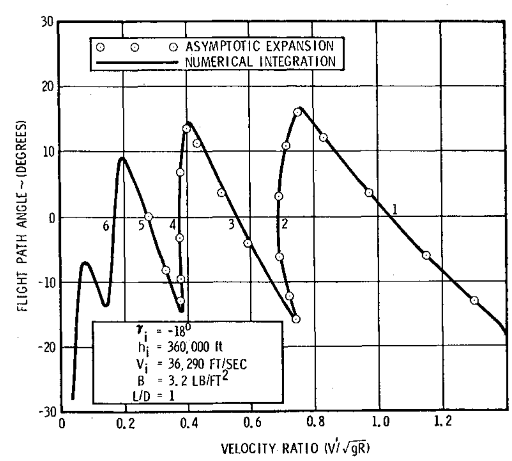

Fig. 2 Flight path angle vs velocity ratio—skipping entry.

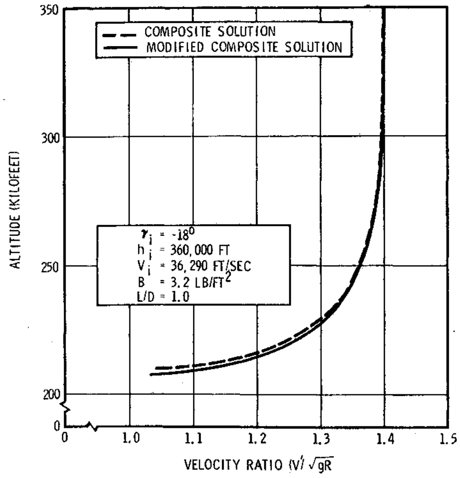

Fig. 3 Altitude vs velocity ratio—first descending arc.

and the exact solution for maximum and minimum points is given in Table 2. Agreement is surprisingly good for this case as the difference between numerical integration and the matched asymptotic solution is only about 3% at the fourth extremal point (the second peak point).

It is important to point out that our numerical solution for the altitude of the first peak results in $\beta h' = 24$ instead of the value of about 15.5 obtained by Loh (Fig. 2 of Ref. 5, Fig. 3 of Ref. 4). The difference is probably caused by the constant gravity assumption of Loh's approximations.

Further insight into the differences between the matched asymptotic approach and Loh's approximate solution may be provided by expanding Loh's solution in powers of the small parameter ϵ and comparing the results with the present approach. Carrying out inner expansion of Eqs. (5) and (7) of Ref. 4 and using the notation of Ref. 1, we obtain

$$s_0 = \cos\gamma_i - \tfrac{1}{2}C_L\rho_i + \tfrac{1}{2}\rho C_L$$
$$V_0^2 = V_i^2 \exp[(C_D/2C_L)(\gamma_0 - \gamma_i)] \tag{8}$$

$$s_1 = (1 - 1/V_0^2)(1 - \rho_i/\rho)[\cos\gamma_i + \tfrac{1}{2}\rho C_L(1 - \rho_i/\rho)] \tag{9}$$

For the purposes of comparison, s_1 is differentiated to yield

$$\frac{ds_1}{dh^*} = s_0 \left\{ -\left(1 - \frac{1}{V_0^2}\right)\left(1 + \frac{\rho_i}{\rho} - \frac{\cos\gamma_i}{s_0}\right) + \frac{\rho C_D}{\sin\gamma_0}\left(1 - \frac{\rho_i}{\rho}\right)\right\} \tag{10}$$

Comparing Eq. (8) with Eqs. (31) and (37) and Eq. (10) with Eq. (30) of Ref. 1, it is evident that inner expansion of Loh's approximate solution of Ref. 4 agrees in functional form with our inner solution to the lowest order. Since Loh's solution embodies the assumptions of constant gravity as well as holding constant a group of terms during integration, his approximation does not agree with our composite solution to any order in the outer region.

In summary, the present paper demonstrates that the composite solution can be applied for calculating skipping entry trajectories and numerically confirms the quality of matched asymptotic solutions. The computational procedure is much simpler than that used by Loh[4,5] and was obtained systematically by use of a well-established asymptotic expansion procedure[6,7] rather than the heuristic approach employed by Loh. In view of the success encountered in applying asymptotic methods to lift controlled atmospheric entry problems,[8] it may be conjectured that the basic approach employed here may be extended to optimum trajectories involving skipping such as the maximum range problem considered in Ref. 9.

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An Optimum Design for the Instability of Cylindrical Shells under Lateral Pressure

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I. Introduction

THE problem of determining the strongest shell for a given weight is an out-growth of the problem treated by Tadjbakhsh and Keller[1] who determined the shape of a column which extremized the buckling load for a given weight and length. Taylor[2] subsequently showed that the governing equation of Ref. 1 for extremizing the eigenvalue can be derived from an energy functional. This method was applied by Huang and Sheu[3] to a thin-walled column under its own weight and by Wu[4] to a circular arch under lateral pressure.

The direct application of these concepts to buckling of two-dimensional structures does not appear to be very fruitful, because of the complexity of the resulting nonlinear partial differential equations. As an alternative procedure, solutions can be generated for a large number of geometries and the optimum chosen by examination. However, this procedure is very inefficient in comparison to a direct determination of the optimum by extremizing the eigenvalue, provided that the mathematical complexities associated with the latter approach can be overcome.

Optimum designs based on a direct extension of Refs. 1, 3, and 4 involve the determination of shape functions which

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